Linear programming

Optimization problem

$$(P)\begin{cases}\text{minimize} & f(\mathbf{x})\\ \text{subject to} & g_1(\mathbf{x})\le b_1\\ & \vdots\\ & g_m(\mathbf{x})\le b_m,\end{cases}$$

where $\mathbf{x}\in\mathbb{R}^n$, $f,g_i:\mathbb{R}^n\to\mathbb{R}$, $b_i\in\mathbb{R}$. Program (P) is *linear* if f,g_i are linear functions. Reformulation:

$$(LP)\begin{cases}\text{minimize} & \mathbf{c}^T\mathbf{x}\\ \text{subject to} & A\mathbf{x}\le\mathbf{b},\end{cases}$$

where $\mathbf{c}\in\mathbb{R}^n$, $\mathbf{b}\in\mathbb{R}^m$, $A\in\mathbb{R}^{m\times n}$. Also maximize, $\ge$, $=$. A program (LP) is efficiently solvable (P-time). Note that $<$ or $>$ are NOT allowed.

History note: 1939 by Kantorovich[1], Dantzig (simplex method).

1: Write the following (LP) in the matrix form.

$$(LP)\begin{cases}\text{minimize} & x+y\\ \text{subject to} & x+2y\le 14\\ & 3x-y\ge 0\\ & x-y\le 2\end{cases}$$

Solution:

$$(LP)\begin{cases}\text{minimize} & (1,1)\cdot\begin{pmatrix}x_1\\x_2\end{pmatrix}\\ \text{subject to} & \begin{pmatrix}1&2\\-3&1\\1&-1\end{pmatrix}\cdot\begin{pmatrix}x_1\\x_2\end{pmatrix}\le\begin{pmatrix}14\\0\\2\end{pmatrix}\end{cases}$$

[1] Full professor at age 22.

2: Diet problem: Formulate as a linear programming problem the following question: *How many apricots (x_1), bananas (x_2) and cucumbers (x_3) does one have to eat to get enough of vitamins A, B, and C while minimizing the cost?*

Need to know: % of recommended daily intake, cost, and weight:

	A	C	K	$	weight
apricots	60	26	6	1.53	155g
bananas	3	33	1	0.37	225g
cucumbers	2	7	12	0.18	133g

Solution:

$$(LP)\begin{cases} \text{minimize} & 1.53x_1 + 0.37x_2 + 0.18x_3 \\ \text{s.t.} & 60x_1 + 3x_2 + 2x_3 \geq 100 \\ & 26x_1 + 33x_2 + 7x_3 \geq 100 \\ & 6x_1 + 1x_2 + 12x_3 \geq 100 \end{cases}$$

Solution: $(x_1, x_2, x_3) = (1.4, 0.3, 7.6)$. The answer is $1.4 \cdot 155g$, $0.3 \cdot 225g$, and $7.6 \cdot 133g$ of apricots, bananas, and cucumbers respectively and the cost is $3.62.

Solution can be obtained using APMonitor. Go to `apmonitor.com`, click Try online and paste the code below.

```
Model fruit
  Variables
    x[1] = 0, >= 0
    x[2] = 0, >= 0
    x[3] = 0, >= 0
  End Variables
  Equations
  minimize 1.53*x[1]+ 0.37*x[2]+0.18*x[3]
  60*x[1]+3*x[2]+2*x[3] >= 100
  26*x[1]+33*x[2]+7*x[3] >= 100
   6*x[1] + x[2] +12*x[3] >= 100
  End Equations
End Model
```

3: Farming: A farmer has 12 acres of land to plant either soybeans or corn. At least 7 acres have to be planted. Planting one acre of soybeans costs $200 and one acre of corn costs $100. Budget for planting is $1500. The sale from one acre of soybeans is $500 and from corn is $300. How many acres of what should be planted to maximize profit?

Linear programming was the biggest invention in mathematics in the last century - if measured by $.

Solution:

$$(LP)\begin{cases}\text{minimize} & (500-200)soy+(300-100)corn\\ \text{s.t.} & 200soy+100corn\le 1500\\ & soy+corn\le 12\\ & soy+corn\ge 7\\ & soy\ge 0\\ & corn\ge 0\end{cases}$$

Solution is 3 acres of soy and 9 acres of corn. Profit is $2700.

APMonitor and Sage writeup of the problem.

```
Model farmer  ! APmonitor
  Variables
    soy  = 0, >= 0
    corn = 0, >= 0
  End Variables
  Equations
  maximize 500*soy + 300*corn  - 200*soy - 100*corn
    200*soy + 100*corn <= 1500
    soy + corn <= 12
    soy+corn >= 7
  End Equations
End Model
```

```
p = MixedIntegerLinearProgram(maximization=True)  # Sage
x = p.new_variable(nonnegative=True)
p.set_objective( 500*x[0] + 300*x[1] - 200*x[0] - 100*x[1])
p.add_constraint( 200*x[0] + 100*x[1] <= 1500 )
p.add_constraint( x[0] + x[1] <= 12 )
p.add_constraint( x[0] + x[1] >= 7 )
print "Profit $", p.solve()
print "Soybeans",p.get_values(x[0]),"acres, Corn",p.get_values(x[1]),"acres"
```

4: 2-Player Zero-Sum Games:

In penalty kicks in soccer (football in World\USA), the kicker (number 7) kicks the ball and usually tries to aim at one of the top corners. The goalie (number 11) tries to guess which corner the kicker kicks and jumps towards one of the corner. If the goalie has a correct guess, there is a very good change for the goalie to catch the ball. If the goalie guesses wrong, it is a goal unless the kicker messes up.

Assume you are the kicker and you know that the goalie has a handicap that if you shoot to the left and the goalie jumps left, there is only 10% chance for you to score but if you kick to the right and the goalie jumps to the right, there is 50% chance of scoring. If the goalie jumps in the opposite direction than your kick, you have 95% chance of scoring. Should you kick the ball to the left or to the right?

If you always kick to the right, the goalie will always jump to the right and you score 0.5 goals per kick. It is better to pick left or right with some probability. What is the best left-right probability subject to the goalie picking his random jumps to counter your strategy as much as possible?

Solution: Lets create a scoring table. In the row, the kicker picks left or right, then goalie picks left or right (not knowing the kicker's pick) and the outcome is in the table.

		goalie	
		left	right
kicker	left	0.1	0.95
	right	0.95	0.5

To formulate this as a linear program, we start with variables ℓ and r. We also add a variable s, which is the expected score (number of goals).

$$(LP)\begin{cases} \text{maximize} & s \\ \text{s.t.} & 0.1\ell + 0.95r \geq s \\ & 0.95\ell + 0.5r \geq s \\ & \ell + r = 1 \\ & \ell \geq 0 \\ & r \geq 0 \end{cases}$$

The solution is approximately $\ell = 0.346$, $r = 0.654$ and $s = 0.6557$. Notice that this randomized strategy gives at least 0.65557 no matter what is the strategy of the goalie.

5: Ropes: We are producing packages of two 15cm ropes and one 20cm rope (say for some kid's game). Suppose we have have 400 times 50cm ropes and 100 times 65cm ropes. How should we cut the ropes to maximize the number of produced packages?

Solution: #15 cm $= A$, #20 cm $= B$,

$$\begin{aligned} 50cm &= 15+15+20 \ldots x_1 \ldots 2A+B \\ &= 20+20 \ldots x_2 \ldots 2B \\ &= 15+15+15 \ldots x_3 \ldots 3A \end{aligned}$$

$$\begin{aligned} 65cm &= 20+20+20 \ldots y_1 \ldots 3B \\ &= 15+15+15+15 \ldots y_2 \ldots 4A \\ &= 20+15+15+15 \ldots y_3 \ldots B+3A \\ &= 20+20+15 \ldots y_4 \ldots 2B+A \end{aligned}$$

$$(LP)\begin{cases} \text{maximize} & p \\ \text{s.t.} & p \le \frac{1}{2}A \\ & p \le B \\ & A = 2x_1 + 3x_3 + 4y_2 + 3y_3 + y_4 \\ & B = x_1 + 2x_2 + 3y_1 + y_3 + 2y_4 \\ & 400 \ge x_1 + x_2 + x_3 \\ & 100 \ge y_1 + y_2 + y_3 \end{cases}$$

Solution:

$$p = 528.5, x_1 = 400, x_2 = 0, x_3 = 0, y_1 = 14.28, y_2 = 0, y_3 = 85.71, y_4 = 0$$

We are missing that x_i, y_j are actually integers! Adding the constraint that the variables are integers result in significantly more difficult problem.

Geometry behind linear programming and basics

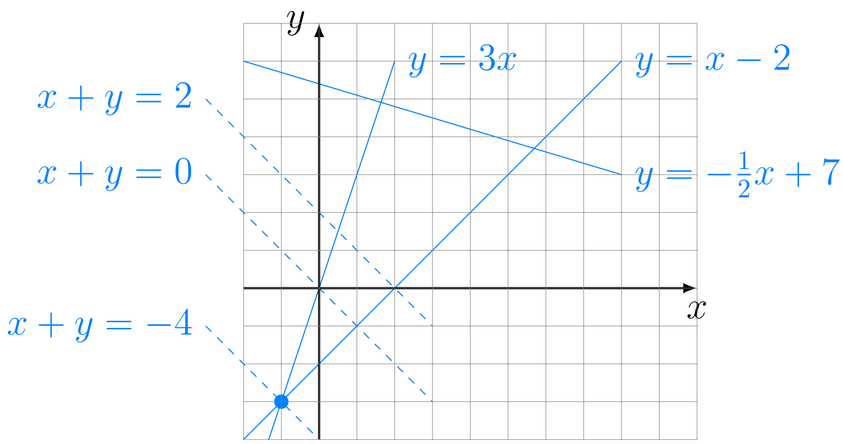
6: Solve the following linear program:

$$(LP)\begin{cases}\text{minimize} & x+y\\ \text{s.t.} & x+2y\le 14\\ & 3x-y\ge 0\\ & x-y\le 2,\end{cases}$$

Hint: Plot points (x,y) that satisfy all constraints and then identify the optimal solution among them.

Solution: equations:

$$y\le -\frac{1}{2}x+7 \qquad y\le 3x \qquad y\ge x-2$$



Optimum $(x,y)=(-1,-3)$, value of objective function is -4.

Recall that a linear program can be written using a matrix $A\in\mathbb{R}^{m\times n}$ and vectors $\mathbf{b}\in\mathbb{R}^m$ and $\mathbf{c},\mathbf{x}\in\mathbb{R}^n$ as

$$(LP)\begin{cases}\text{minimize} & \mathbf{c}^T\mathbf{x}\\ \text{s.t.} & A\mathbf{x}\le\mathbf{b}\end{cases}$$

Basic linear programming definitions:

- *feasible solution* is vector $\mathbf{x}$ such that $A\mathbf{x}\le\mathbf{b}$. In other words, a point satisfying all the constraints.
- *a set of feasible solutions*
- an *optimal solution* is a feasible solution that is maximizing/minimizing the objective function.

7: What shape is the set of feasible solutions?

Solution: In the example above, a polygon. In general, polyhedra ("unbounded polytope"). In 2D we could say it is an intersection of halfplanes (halfspaces in xD).

8: What shape is the set of optimal solutions?

Solution: In the example above, it is a point. But it can be also a line. Or a special case, where it can be any feasible point (maybe you just want to know if a feasible solution exists).

9: Construct a linear program that has no feasible solution.

Solution:

$$(LP)\begin{cases}\text{minimize} & x\\ \text{s.t.} & x \le 3\\ & x \ge 4\end{cases}$$

10: Construct a linear program that has a feasible solution but does not have an optimal solution.

Solution:

$$(LP)\begin{cases}\text{maximize} & x\\ \text{s.t.} & x \ge 4\end{cases}$$

11: Construct a linear program that has more than one optimal solution.

Solution:

$$(LP)\begin{cases}\text{minimize} & x\\ \text{s.t.} & x \le 3\\ & y \ge 0\\ & y \le 2\end{cases}$$

Introduction to the Duality for Linear Programming

Let (P) be

$$(P)\begin{cases}\text{maximize} & 2x_1 + 3x_2\\ \text{s.t.} & 4x_1 + 8x_2 \le 12\\ & 2x_1 + x_2 \le 3\\ & 3x_1 + 2x_2 \le 4\\ & x_1 \ge 0\\ & x_2 \ge 0\end{cases}$$

12: Without solving (P) itself, is it possible to provide an upper bound on the value of (P) by using equation $4x_1 + 8x_2 \le 12$?

Solution: Yes - easily:

$$2x_1 + 3x_4 \le 4x_1 + 8x_2 \le 12$$

so the maximum is at most 12. We can even improve it by

$$2x_1 + 3x_4 \le \frac{1}{2}(4x_1 + 8x_2) \le 6.$$

This gives a maximum of at most 6.

13: Without solving (P), is it possible to provide an upper bound on the value of (P) using equations $4x_1 + 8x_2 \le 12$ and $2x_1 + x_2 \le 3$? *Hint: sum them*

Solution:

$$\begin{array}{rr} & 4x_1 + 8x_2 \le 12\\ & 2x_1 + x_2 \le 3\\ + & \overline{}\\ & 6x_1 + 9x_2 \le 15\end{array}$$

Now we get $2x_1 + 3x_4 \le \frac{1}{3}(6x_1 + 9x_2) \le 5$.

14: Without solving (P), how would you try to find the combination of constraints that provides the best upper bound? (solution might be another linear program, call it (D))

Solution: We try to combine the three constraints (not the non-negativity constraints) and obtain an upper bound. Say the first constraints is multiplied by y_1, the second by y_2 and third by y_3.

So we have a combination of

$$\begin{aligned} &y_1 \cdot (4x_1 + 8x_2 \le 12)\\ &y_2 \cdot (2x_1 + x_2 \le 3)\\ &y_3 \cdot (3x_1 + 2x_2 \le 4)\end{aligned}$$

What else does y_i satisfy? If $y_i < 0$, the inequality is reversed, so $y_i \geq 0$. We need the left hand sides to be at least $2x_1 + 3x_4$, hence

$$y_1 \cdot 4x_1 + y_2 \cdot 2x_1 + y_3 \cdot 3x_1 \geq 2x_1$$
$$y_1 \cdot 8x_2 + y_2 \cdot x_2 + y_3 \cdot 2x_2 \geq 3x_2$$

Next, we want to minimize the right hand side, which is $12y_1 + 3y_2 + 4y_3$. It gives a linear program (D):

$$(D)\begin{cases} \text{minimize} & 12y_1 + 3y_2 + 4y_3 \\ \text{s.t.} & 4y_1 + 2y_2 + 3y_3 \geq 2 \\ & 8y_1 + y_2 + 2y_3 \geq 3 \\ & y_1, y_2, y_2 \geq 0 \end{cases}$$

- (D) gives an upper bound on (P)
- (P) gives a lower bound on (D)

15: Are solutions $\mathbf{x} = (\frac{1}{2}, \frac{5}{4})$ of (P) and $\mathbf{y} = (\frac{5}{16}, 0, \frac{1}{4})$ for (D) optimal solutions?

Solution: Yes! They are optimal solutions because they satisfy all constraints and values of the objective functions are the same.

16: Find the dual program (D) to

$$(P)\begin{cases} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{cases}$$

Solution:

$$(D)\begin{cases} \text{minimize} & \mathbf{b}^T\mathbf{y} \\ \text{s.t.} & A^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{cases}$$

17: Find the dual program (D) to

$$(P)\begin{cases} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{cases}$$

Solution: We first rewrite $A\mathbf{x} = \mathbf{b}$ as $A\mathbf{x} \geq \mathbf{b}$ and $-A\mathbf{x} \geq -\mathbf{b}$
Then we get

$$(P)\begin{cases} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \geq \mathbf{b} \\ & -A\mathbf{x} \geq -\mathbf{b} \\ & \mathbf{x} \geq 0 \end{cases} \qquad (D)\begin{cases} \text{maximize} & \mathbf{b}, -\mathbf{b}^T(\mathbf{u}, \mathbf{v}) \\ \text{s.t.} & A^T\mathbf{u} - A^T\mathbf{v} \leq \mathbf{c} \\ & \mathbf{u}, \mathbf{v} \geq 0 \end{cases}$$

Let $\mathbf{y} = \mathbf{u} - \mathbf{v}$. Then we can write

$$(D) \begin{cases} \text{maximize} & \mathbf{b}^T\mathbf{y} \\ \text{s.t.} & A^T\mathbf{y} \le \mathbf{c} \end{cases}$$

Note that $\mathbf{y}$ can be negative.

Dualization for everyone:
$A \in \mathbb{R}^{m\times n}, \mathbf{c} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^m$

	primal	dual
variables	$x_1, \ldots, x_n$	$y_1, \ldots, y_m$
matrix	A	A^T
right hand	$\mathbf{b}$	$\mathbf{c}$
objective	max $\mathbf{c}^T\mathbf{x}$	min $\mathbf{b}^T\mathbf{y}$
constraint	ith constrain $\le$	$y_i \ge 0$
	ith constrain $\ge$	$y_i \le 0$
	ith constrain $=$	$y_i \in \mathbb{R}$
	$x_i \ge 0$	ith constrain $\ge$
	$x_i \le 0$	ith constrain $\le$
	$x_i \in \mathbb{R}$	ith constrain $=$

Strong Duality Theorem
For the linear programs

$$\text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} \le \mathbf{b} \text{ and } \mathbf{x} \ge \mathbf{0} \qquad (P)$$

and

$$\text{minimize } \mathbf{b}^T\mathbf{y} \text{ subject to } A^T\mathbf{y} \ge \mathbf{c} \text{ and } \mathbf{y} \ge \mathbf{0} \qquad (D)$$

exactly one of the following possibilities occurs:

1. Neither (P) nor (D) has a feasible solution.
2. (P) is unbounded and (D) has no feasible solution.
3. (P) has no feasible solution and (D) is unbounded.
4. Both (P) and (D) have a feasible solution. Then both have an optimal solution, and if $\mathbf{x}^\star$ is an optimal solution of (P) and $\mathbf{y}^\star$ is an optimal solution of (D), then

   $$\mathbf{c}^T\mathbf{x}^\star = \mathbf{b}^T\mathbf{y}^\star.$$

   That is, *the maximum of* (P) *equals the minimum of* (D).